DYNAMIC CONTACT PROBLEM OF STEADY VIBRATIONS OF AN ELASTIC HALF-SPACE

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Two problems are treated with a common formulation: the effect of a sinusoidal load (with a time factor of the form $\exp -i\omega t$) transmitted through a weightless (and inertialess) rigid punch of circular planform lying on the homogeneous isotropic half-space ($z \geqslant 0$). The vibrations are steady-state, and the effect of the static load is not considered, it being assumed that its magnitude is sufficient to maintain contact in the case where the punch is not joined to the half-space. For brevity, henceforth when we speak of vectors (or their projections), we shall mean their amplitudes. The actual values are obtained by multiplying by $\exp(-i\omega t)$. In the fundamental mixed problem it is required to find, for a known distribution of the displacement vector $\mathbf{w}(w, u_x, u_y)$ in the region of contact Ω, the distribution of the force vector $\mathbf{p}\{p, t_x, t_y\}$ (p is the pressure, t_x, t_y the tangential forces) in this region. In particular, if there is bonding, $u_x = u_y = 0$. In the other problem (a punch without friction and bonding) we have $t_x = t_y = 0$, and it is required to find only $p = p(x, y)$ for a known $w = w(x, y)$ in the region Ω.

A unified treatment of these problems is given in deriving the integral equations and developing an approximate method for their solution (which is applied to the second problem).

1. The proposed method consists of the following. The system of equations of the fundamental mixed problem (in vector notation, of course) takes the form

$$\mathbf{w}(r) = \iint_{\Omega} K(\mathbf{r} - \mathbf{r}')\, \mathbf{p}(\mathbf{r}')\, dS, \qquad r \in \Omega \tag{1.1}$$

Here $K(\mathbf{r} - \mathbf{r}')$ is the difference matrix kernel. The columns (K^0, K^1, K^2) of the matrix $K(\mathbf{r})$ represent the displacement vectors due to the effects of concentrated unit forces applied at the origin and directed along the axes z, x, y, respectively. It is therefore clear that the kernel $K(\mathbf{r})$ is singular (with a weak singularity). By separating out the singularity it is possible to reduce (1.1) to a system of equations of the second kind. It will subsequently be shown that the terms in $K(\mathbf{r})$ containing singularities correspond to the static problem. Therefore if the solution of the static problem is known, then the transition to a system of equations of the second kind can in fact be carried out. In order to find $K(\mathbf{r})$ we solve the following system of Lamé equations (in rectangular coordinates)

$$(\lambda + \mu) \nabla \operatorname{div} \mathbf{w} + \mu \Delta \mathbf{w} + \rho\omega^2 \mathbf{w} = 0 \qquad (\rho \text{ is the density of the elastic medium}) \tag{1.2}$$

for the conditions

$$(\sigma_z, \tau_{zx}, \tau_{zy}) = (P, T_x, T_y)\, \delta(x)\, \delta(y) \tag{1.3}$$

The calculations lead to the following result:

$$\mathbf{w} = K\mathbf{P}, \qquad \mathbf{P} = \begin{bmatrix} P \\ T_x \\ T_y \end{bmatrix}, \qquad K = \frac{1}{4\pi^2\mu} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\alpha\, d\beta \exp i(\alpha x + \beta y) \times$$

$$\times \begin{bmatrix} E(\gamma^2), & i\alpha\Theta(\gamma^2), & i\beta\Theta(\gamma^2) \\ -i\alpha\Theta(\gamma^2), & {}^1/_2 H(\gamma) + {}^1/_2(\alpha^2 - \beta^2) Z(\gamma^2), & \alpha\beta Z(\gamma^2) \\ -i\beta\Theta(\gamma^2), & \alpha\beta Z(\gamma^2), & {}^1/_2 H(\gamma^2) - {}^1/_2(\alpha^2 - \beta^2) Z(\gamma^2) \end{bmatrix} \tag{1.4}$$

$$(\gamma = \sqrt{\alpha^2 + \beta^2})$$

The functions E, Θ, H, Z of the argument γ^2 appearing here represent the quotients obtained by dividing the functions E_*, Θ_*, H_*, Z_*, whose expressions are given below, by the product $(\gamma^2 - k_2^2)^{1/2} R(\gamma^2)$, where R is the Rayleigh function

$$R(\gamma^2) = (2\gamma^2 - k_2^2) - 4\gamma^2 (\gamma^2 - k_1^2)^{1/2} (\gamma^2 - k_2^2)^{1/2} \tag{1.5}$$

(k_1 and k_2 are wave numbers).

The expressions for E_*, Θ_*, H_*, Z_* are

$$E_*(\gamma^2) = -k_1^2 (\gamma^2 - k_1^2)^{1/2} (\gamma^2 - k_2^2)^{1/2}, \quad H_*(\gamma^2) = R(\gamma^2) - k_2^2 (\gamma^2 - k_2^2) \tag{1.6}$$

$$\Theta_*(\gamma^2) = (2\gamma^2 - k_2^2)(\gamma^2 - k_2^2)^{1/2} - 2(\gamma^2 - k_2^2)(\gamma^2 - k_1^2)^{1/2}$$

$$\gamma^2 Z_*(\gamma^2) = -R(\gamma^2) - k_2^2(\gamma^2 - k_2^2) \quad (k_1 = \omega\sqrt{\rho/(\lambda + 2\mu)},\ k_2 = \omega\sqrt{\rho/\mu})$$

The radicals appearing in these formulas are single-valued functions of γ on the Riemann surface consisting of four sheets, corresponding to the four sign combinations. The sheets are joined along a cut which is drawn in an appropriate manner. We will use a set of cuts which was applied by A. Sommerfeld to a problem of radio-wave propagation [1] (p. 945). The double Fourier integral in (1.4) may be reduced to a single integral by transforming to polar coordinates in the α, β plane and performing the integration over the angular variable. In this calculation and in the sequel it is convenient to deal not directly with the vector displacements and forces, but rather with the following complex combinations of them

$$w_1 = u + iv, \quad w_2 = u - iv, \quad t_1 = t_x + it_y, \quad t_2 = t_x - it_y \tag{1.7}$$

Then

$$\begin{bmatrix} w \\ w_1 \\ w_2 \end{bmatrix} = \Lambda \begin{bmatrix} p \\ {}^1/_2 t_1 \\ {}^1/_2 t_2 \end{bmatrix}, \qquad \Lambda = \frac{1}{4\pi^2\mu} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\alpha\, d\beta \exp(\alpha x + \beta y) \times$$

$$\times \begin{bmatrix} E(\gamma^2), & (\beta + i\alpha)\Theta(\gamma^2), & (-\beta + i\alpha)\Theta(\gamma^2) \\ (\beta - i\alpha)\Theta(\gamma^2), & H(\gamma^2), & -(\beta - i\alpha)^2 Z(\gamma^2) \\ (-\beta - i\alpha)\Theta(\gamma^2), & -(\beta + i\alpha)^2 Z(\gamma^2), & H(\gamma^2) \end{bmatrix} \tag{1.8}$$

Expressing the integrals in polar coordinates, we obtain

$$\Lambda(x, y) = \frac{1}{2\pi\mu}\int_0^\infty \gamma d\gamma \begin{bmatrix} E^\circ(\gamma^2) J_0(\gamma r), & -\frac{\bar{z}}{r}\gamma\,\Theta(\gamma^2) J_1(\gamma r), & \frac{z}{r}\gamma\Theta(\gamma^2) J_1(\gamma r) \\ \frac{z}{r}\gamma\Theta(\gamma^2) J_1(\gamma r), & H(\gamma^2) J_0(\gamma r), & -\frac{z^2}{r^2}\gamma^2 Z(\gamma^2) J_2(\gamma r) \\ \frac{\bar{z}}{r}\gamma\Theta(\gamma^2) J_1(\gamma r), & -\frac{\bar{z}^2}{r^2}\gamma^2 Z(\gamma^2) J_2(\gamma r), & H(\gamma^2) J_0(\gamma r) \end{bmatrix} \tag{1.9}$$

where

$$z = x + iy, \quad \bar{z} = x - iy, \quad r = |z| = \sqrt{x^2 + y^2}$$

There is no danger of confusion with the spatial coordinate z, since the latter never appears in the formulas. To calculate the asymptote of this kernel, one may, as shown in [1], transform to the interval $(-\infty, +\infty)$ in the Hankel functions $H^{(1)}(\gamma r)$, using the relation of the circuits. A formula for $\Lambda(x, y)$ is then obtained which differs from (1.9) by the multiplier in front of the integral: $1/4\pi\mu$ instead of $1/2\pi\mu$, whereas the Hankel function $H^{(1)}(\gamma r)$ with the corresponding indices will appear in the integrands. We will not write out this formula, since we do not aim to present a complete detailed treatment of the asymptote. We note only that for a concentrated normal force $\mathbf{P} = (P, 0, 0)$ the asymptotic displacement was investigated by Lamb [2]. We also note that in calculating the integrals the original path of integration $(-\infty, +\infty)$ is deformed, remaining in the upper half-plane. As a consequence of this the integral includes the contribution of the pole associated with the root of the equation $R(\gamma^2) = 0$ in the right-hand quadrant. In calculating the integrals appearing in (1.9), however, this pole is bypassed from above; thus each of these integrals is equal to the principal value minus the product of $i\pi$ and the residue corresponding to this pole. Hence the vector equation (1.1) reduces to

$$\begin{bmatrix} w \\ w_1 \\ w_2 \end{bmatrix}(x, y) = \iint_\Omega \Lambda(x - \xi, y - \eta)\begin{bmatrix} p \\ {}^1/_2 t_1 \\ {}^1/_2 t_2 \end{bmatrix}(\xi, \eta)\, d\xi\, d\eta \tag{1.10}$$

2. For arbitrarily shaped regions Ω, there is no effective analytical method of solving these two-dimensional equations. A simplification is possible for the case of a circular region, however. With this aim we expand the force vector (p, t_r, t_θ) in a Fourier series in $e^{im\theta}$

$$(p, t_r, t_\theta) = \sum_{-\infty}^{+\infty} (f_m(r), g_m(r), h_m(r))\, e^{im\theta} \tag{2.1}$$

For each of the components $t_{rm} = g_m(r)\, e^{im\theta}$, $t_{\theta m} = h_m(r)\, e^{im\theta}$ we have

$$t_{1m} = [g_m(r) + ih_m(r)]\, e^{i(m+1)\theta}, \qquad t_{2m} = [g_m(r) - ih_m(r)]\, e^{i(m-1)\theta} \tag{2.2}$$

We note further that

$$w_1 = u + iv = (u_r + iu_\theta)\, e^{i\theta}, \qquad w_2 = u - iv = (u_r - iu_\theta)\, e^{-i\theta}$$

Introducing these values in (1.10), where z should be replaced by $z - \zeta = re^{i\theta} - \rho e^{i\psi}$ etc. we perform the integration over the angle ψ with the help of Graf's addition theorem [3] (p. 392). We give only the result of the calculation

$$2\mu\begin{bmatrix} 2w \\ u_r + iu_\theta \\ u_r - iu_\theta \end{bmatrix}_m (r) = \int_0^a \Lambda^{(m)}(r,\rho)\begin{bmatrix} 2f_m \\ g_m + ih_m \\ g_m - ih_m \end{bmatrix}(\rho)\,\rho d\rho \tag{2.3}$$

Here the index m denotes the Fourier coefficient (of $e^{im\theta}$). The kernel $\Lambda^{(m)}$ is a matrix with the following elements:

$$\Lambda_{00}^{(m)} = 2\int_0^\infty E(\gamma^2) J_m(\gamma r) J_m(\gamma\rho)\gamma\, d\gamma, \qquad \Lambda_{20}^{(m)} = \int_0^\infty \Theta(\gamma^2) J_{m-1}(\gamma r) J_m(\gamma\rho)\gamma^2 d\gamma$$

$$\Lambda_{01}^{(m)} = \int_0^\infty \Theta(\gamma^2) J_{m+1}(\gamma\rho) J_m(\gamma r)\gamma^2 d\gamma, \qquad \Lambda_{11}^{(m)} = \int_0^\infty H(\gamma^2) J_{m+1}(\gamma r) J_{m+1}(\gamma\rho)\gamma\, d\gamma$$

$$\Lambda_{02}^{(m)} = \int_0^\infty \Theta(\gamma^2) J_{m-1}(\gamma\rho) J_m(\gamma r)\gamma^2 d\gamma, \qquad \Lambda_{22}^{(m)} = \int_0^\infty H(\gamma^2) J_{m-1}(\gamma r) J_{m-1}(\gamma\rho)\gamma\, d\gamma$$

$$\Lambda_{10}^{(m)} = \int_0^\infty \Theta(\gamma^2) J_{m+1}(\gamma r) J_m(\gamma\rho)\gamma^2 d\gamma, \qquad \Lambda_{12}^{(m)} = -\int_0^\infty Z(\gamma^2) J_{m+1}(\gamma r) J_{m+1}(\gamma\rho)\gamma^3 d\gamma$$

$$\Lambda_{21}^{(m)} = -\int_0^a Z(\gamma^2) J_{m+1}(\gamma\rho) J_{m+1}(\gamma r)\gamma^3 d\gamma \tag{2.4}$$

The remark made at the end of the first section holds for the calculation of these integrals. It would have been possible here to express $(w, u_r, u_\theta)_m$ directly in terms of $(f, g, h)_m$, but the formulas for the elements of the kernel would then have been complicated. The notation would be even more complicated if we went from the complex form of the Fourier series to the real form. Only in the case $m = 0$ is there a simplification

$$\begin{bmatrix} w \\ u_r \end{bmatrix}(r) = \frac{1}{\mu}\int_0^a \begin{bmatrix} K_{00}, & K_{01} \\ K_{10}, & K_{11} \end{bmatrix}(r,\rho)\begin{bmatrix} f_0 \\ g_0 \end{bmatrix}(\rho)\,\rho d\rho, \qquad u_\theta(r) = \frac{1}{\mu}\int_0^a K_{22}(r,\rho)\, h(\rho)\, d\rho \tag{2.5}$$

(the subscript $m = 0$ is omitted here). In these equations

$$K_{00} = -\int_0^\infty \frac{k_2^2(\gamma^2 - k_1^2)^{1/2}}{R(\gamma^2)} J_0(\gamma r) J_0(\gamma\rho)\gamma\, d\gamma$$

$$K_{11} = -\int_0^\infty \frac{k_2^2(\gamma^2 - k_2^2)}{R(\gamma^2)} J_1(\gamma r) J_1(\gamma\rho)\gamma\, d\gamma, \qquad K_{01} = \int_0^\infty \gamma^2\Theta(\gamma^2) J_0(\gamma r) J_1(\gamma\rho)\, d\gamma$$

$$K_{10} = \int_0^\infty \gamma^2\Theta(\gamma^2) J_0(\gamma\rho) J_1(\gamma r)\, d\gamma, \qquad K_{22} = \int_0^\infty \frac{J_1(\gamma r) J_1(\gamma\rho)}{(\gamma^2 - k_2^2)^{1/2}}\gamma\, d\gamma \tag{2.6}$$

Equations (2.5) may be considered as the equations of the axisymmetric problem, where the first of equations (2.5) describes the indentation of the punch, while the second describes torsion (the Reissner-Sagoci problem). The resolution of the system of integral equations into two independent systems corresponds to the decomposition of the system of Lamé equations in the axisymmetric case. The vanishing of $R(\gamma^2)$ in the second of equations (2.5), which is clear from the physical point of view, depends on the fact that as a consequence of formula (1.6) we have

$$H(\gamma^2) - \gamma^2 Z(\gamma^2) = 2(\gamma^2 - k_2^2)^{1/2}$$

The solution of the system (2.3) even in the simplest axisymmetric case (2.5) turns on the absence of any simple exact solution of the static problem. Cumulative approximate quadratures will unavoidably reduce the accuracy of any numerical algorithm. Therefore we will restrict the description of such an algorithm to the problem of indentation of a punch without friction and bonding.

3. In the case $g_m = h_m = 0$ we have

$$w_m(r) = \int_0^a f_m(\rho)\,\rho d\rho \int_0^\infty E(\gamma)\,J_m(\gamma r)\,J_m(\gamma\rho)\,\gamma d\gamma \tag{3.1}$$

It is clear that in this equation $w_m(r)$, $f_m(r)$ may be considered to be the coefficients of a real Fourier series, i.e. as the coefficients of $\cos m\theta$ and $\sin m\theta$.

This equation may also be obtained directly from the results of Lamb [2] with the help of the addition theorem. The axisymmetric problem without friction and bonding was effectively solved by N.M. Borodachev [4]. In that paper Hankel transforms were used to reduce the problem to a pair of integral equations. The pair of integral equations were then reduced to an equation of the second kind (by the method of N.N. Lebedev) for a certain auxiliary function. Calculation of the pressure required additional numerical quadratures. The method developed below differs from Borodachev's method in that only the single equation (3.1) is used, which contains directly the sought-for pressure through the unknown functions $f_m(r)$. Equation (3.1) is transformed directly to an equation of the second kind by separating out the singularities of the kernel.

In order to simplify the subsequent calculations we change to dimensionless quantities by setting

$$r = ax,\quad \rho = a\xi,\quad k_2 = \alpha / a,\quad h = k_1 / k_2\quad f_m(r) = \mu\Pi_m(x),\quad w_m(r) = a\Phi_m(x) \tag{3.2}$$

Making the further substitution $\gamma = k_2 s$ inside the integral (3.1), we obtain

$$\Phi_m(x) = \int_0^1 \Pi_m(\xi)\,\xi d\xi \int_0^\infty \frac{-\alpha\sqrt{s^2 - h^2}}{F(s)}\,sJ_m(\alpha xs)\,J_m(\alpha\xi s)\,ds$$
$$F(s) = (2s^2 - 1) - 4s^2(s^2 - 1)^{1/2}(s^2 - h^2)^{1/2} \tag{3.3}$$

In the neighbourhood of $s = \infty$ we have

$$\frac{-\alpha s\sqrt{s^2 - h^2}}{F(s)} = \frac{\alpha}{2(1 - h^2)} - G(s) \qquad (G(\infty) = 0) \tag{3.4}$$

Here G is a regular function. Equation (3.3) takes the form

$$\frac{1}{2(1 - h^2)} \int_0^1 \Pi_m(\xi)\,\xi d\xi \int_0^\infty J_m(x\lambda)\,J_m(\xi\lambda)\,d\lambda =$$
$$= \Phi_m(x) + \int_0^1 \Pi_m(\xi)\,\xi d\xi \int_0^\infty J_m(\alpha xs)\,J_m(\alpha\xi s)\,G(s)\,ds \tag{3.5}$$

The kernel on the left-hand side is a special case of the Weber-Shafkhaitlin integral

$$W_{p,q}^{(r)}(x, \xi) = \int_0^\infty J_p(x\lambda) J_q(\xi\lambda) \lambda^r d\lambda$$

For the equation

$$\int_0^1 \varphi(\xi) W_{p,q}^{(r)}(x, \xi) d\xi = f(x)$$

the inversion formula [5] is known

$$\varphi(x) = \frac{-2^{1-r} x^q}{\Gamma(1/2(1+r+p+q))\Gamma(1/2(1+r+q-p))} \times$$
$$\times \frac{d}{dx}\int_x^1 \frac{t^{1-r-p-q}dt}{(t^2-x^2)^{1/2(1-r+q-p)}} \frac{d}{dt}\int_0^t \frac{u^{1+p} f(u) du}{(t^2-u^2)^{1/2(1-r+p-q)}}$$

Substituting this formula into (3.5), we obtain an equation of the second kind

$$\Pi_m(x) = \Psi_m(x) + \int_0^1 K_m(x, y) \Pi_m(y) dy \qquad (3.6)$$

in which

$$\Psi_m(x) = -\frac{4(1-h^2) x^{m-1}}{\Gamma(m+1/2)\Gamma(1/2)} \frac{d}{dx}\left\{\int_x^1 \frac{t^{1-2m} dt}{(t^2-x^2)^{1/2}} \frac{d}{dt}\int_0^t \frac{u^{1+m}\Phi_m(u)}{(t^2-u^2)^{1/2}} du\right\} \qquad (3.7)$$

$$K_m(x, y) = \frac{2(1-h^2)\sqrt{2}\, x^{m-1}}{\Gamma(m+1/2)\Gamma(1/2)} \frac{d}{dx}\int_x^1 \frac{t^{3/2-m}}{\sqrt{t^2-x^2}}\left\{\int_0^\infty G(s) J_m(\alpha xs) J_{m-1/2}(\alpha st)(\alpha s)^{1/2} ds\right\}dt$$

$$(3.8)$$

4. In the particular case of a plane inclined punch we have

$$w = w_0 + \beta r\cos\theta = a(W_0 + \beta x\cos\theta)$$

In this case

$$\Phi = W_0, \qquad \Phi_1 = \beta x$$ (β is the angle of inclination of the punch)

Here it is convenient to transform to the new unknown functions

$$H_0(x) = \Pi_0(x)(1-x^2)^{1/2}, \qquad H_1(x) = \Pi_1(x)(1-x^2)^{1/2}$$

Noting that $1 - h^2 = 1/2(1-\nu)$, we obtain the equations

$$H_0(x) = \frac{2W_0}{\pi(1-\nu)} - \frac{2}{\pi(1-\nu)}\int_0^1 K_0(x, y) H_0(y) dy \qquad (4.1)$$

$$H_1(x) = \frac{2a\beta}{\pi(1-\nu)} 4x - \frac{2}{\pi(1-\nu)}\int_0^1 K_1(x, y) H_1(y) dy \qquad (4.2)$$

with kernels equal, respectively, to

$$K_0(x, y) = \frac{y}{x}\left(\frac{1-x^2}{1-y^2}\right)^{1/2} \frac{d}{dx}\int_x^1 \frac{tdt}{\sqrt{t^2-x^2}}\int_0^\infty G(\sigma) J_0(\alpha y\sigma)\cos(\alpha t\sigma)\, d\sigma \qquad (4.3)$$

$$K_1(x, y) = y\left(\frac{1-x^2}{1-y^2}\right)^{1/2} \frac{d}{dx}\int_x^1 \frac{dt}{\sqrt{t^2-x^2}}\int_0^\infty G(\sigma) J_1(\alpha y\sigma)\sin(\alpha t\sigma)\, d\sigma \qquad (4.4)$$

We first consider equation (4.1) with the kernel (4.3). The zeroth approximation to the solution of this equation is

$$H_0^{(0)} = \frac{2W_0}{\pi(1-\nu)}$$

This is related to the solution of the corresponding static problem

$$p_0^{(0)}(r) = \frac{2\mu W_0}{\pi(1-\nu)\, a\sqrt{a^2-r^2}}$$

In order to find the first approximation we calculate the integral

$$\int_0^1 K_0(x, y)\, dy = \int_0^1 \frac{y}{x}\left(\frac{1-x^2}{1-y^2}\right)^{1/2}\left\{\frac{d}{dx}\int_x^1 \frac{tdt}{\sqrt{t^2-x^2}}\int_0^\infty G(\sigma) J_0(\alpha y\sigma)\cos(\alpha t\sigma)\, d\sigma\right\} dy$$

Changing the order of integration, we obtain

$$\int_0^1 K_0(x, y)\, dy = \frac{\sqrt{1-x^2}}{x}\frac{d}{dx}\int_x^1 \frac{tdt}{\sqrt{t^2-x^2}}\left\{\int_0^\infty G(\sigma)\cos(\alpha+\sigma)\frac{\sin\alpha\sigma}{\alpha\sigma}\, d\sigma\right\} \qquad (4.5)$$

To calculate the integral

$$I \equiv \int_0^\infty G(\sigma)\cos(\alpha t\sigma)\frac{\sin\alpha\sigma}{\alpha\sigma}\, d\sigma$$

we use the theory of residues. We have

$$I = \left[\pi c(1-\cos\alpha\sigma_0)\cos\alpha t\sigma_0 + \int_0^1 V(\sigma)(1-\cos\alpha\sigma)\cos(\alpha t\sigma)\, d\sigma\right] +$$

$$+\, i\left[\pi c\sin\alpha\sigma_0\cos\alpha t\sigma_0 + \int_0^1 V(\sigma)\sin\alpha\sigma\cos(\alpha t\sigma)\, d\sigma\right]$$

$$\left(c = -\frac{1}{\alpha\sigma_0}\operatorname{res} G(\sigma)\Big|_{\sigma=\sigma_0},\quad V(\sigma) = \frac{\operatorname{Im} G(\sigma)}{\alpha\sigma}\right)$$

where σ_0 is the root of the Rayleigh equation $F(\sigma) = 0$.

Now the function $I = I(\alpha, t)$ may be expanded in a rapidly converging series in even powers of t, whose (complex) coefficients, which depend on α, may be expanded in series with respect to this parameter

$$I(\alpha, t) = \sum_{m=0}^\infty [P_m(\alpha) + iQ_m(\alpha)]\, t^{2m}, \qquad P_m(\alpha) = \frac{(-1)^m \alpha^{2m}}{(2m)!}\sum_{k=1}^\infty a_k^{(m)}\alpha^{2k} \quad (4.6)$$

$$Q_m(\alpha) = \frac{(-1)^m \alpha^{2m}}{(2m)!} \sum_{k=1}^{\infty} b_k^{(m)} \alpha^{2k+1}, \qquad a_k^{(m)} = \frac{(-1)^{k-1}}{(2k)!} [\pi c \sigma_0^{2m+2k} + q_{2m+2k}]$$

$$b_k^{(m)} = \frac{(-1)^{k-1}}{(2k-1)!} [\pi c \sigma_0^{2m+2k-1} + q_{2m+2k-1}], \qquad q_n \equiv \int_0^1 V(\sigma) \sigma^n d\sigma \tag{4.7}$$

Substituting for $I(\alpha, t)$ in (4.5) its expansion (4.6) and integrating, we have

$$\int_0^1 K_0(x, y)\, dy = \sum_{m=0}^{\infty} [P_m(\alpha) + iQ_m(\alpha)] M_m(x)$$

$$\left(M_m(x) = \frac{\sqrt{1-x^2}}{x} \frac{d}{dx} \int_x^1 \frac{t^{2m+1}}{\sqrt{t^2 - x^2}} dt = \frac{d}{du} \int_0^{\frac{1}{2}\pi} u \sin\varphi\, (1 - u^2 \sin^2\varphi)^m d\varphi\right) \tag{4.8}$$

$$(\sqrt{1-x^2} = u, \quad \sqrt{1-t^2} = \tau)$$

The polynomials $M_m(x)$ are of even powers in u and are consequently even-power polynomials in x. We give the first few polynomials

$$M_0(x) = 1, \qquad M_1(x) = 2u^2 - 1, \qquad M_2(x) = {}^8/_3 u^4 + 4u^2 - 1$$

$$M_3(x) = {}^{16}/_5 u^6 - 8u^4 + 6u^2 - 1 \qquad (u^2 = 1 - x^2)$$

It may be shown that $|M_m(x)| < \text{const} \sqrt{m}$, and consequently the series on the right-hand side of equation (4.8) is majorized by the convergent series

$$\sum_{m=0}^{\infty} \frac{\sqrt{m}}{(2m)!} (\alpha_0)^{2m}$$

We give below the results of calculating the coefficients $P_m^{(\alpha)}$ and $Q_m^{(\alpha)}$ for $m = 0, 1, 2, 3, 4$, accurate up to terms in α^{10} for the case $h^2 = {}^1/_3$ ($\lambda = \mu$, $\nu = {}^1/_4$). In this case $\sigma_0 = 1.0877$, $\pi c = 1.0248$. Using the formulas (4.7) and a table of values of q_n for $h^2 = {}^1/_3$, taken from reference [4], we find

$$P_0(\alpha) = 1.0068\alpha^2 - 0.07913\alpha^4 + 0.002665\alpha^6 - 0.00005433\alpha^8 + \ldots$$

$$Q_0(\alpha) = 2.3243\alpha - 0.3186\alpha^3 + 0.01622\alpha^5 - 0.0004206\alpha^7 + \ldots$$

$$P_1(\alpha) = -\frac{\alpha^2}{2!} [0.9495\alpha^2 - 0.07986\alpha^4 + 0.000312\alpha^6 - \ldots]$$

$$Q_1(\alpha) = -\frac{\alpha^2}{2} [1.9113\alpha - 0.3244\alpha^3 + 0.01766\alpha^5 - 0.0004761\alpha^7 + \ldots]$$

$$P_2(\alpha) = \frac{\alpha^4}{4!} [0.9583\alpha^2 - 0.09361\alpha^4 + 0.003560\alpha^6 - \ldots$$

$$Q_2(\alpha) = \frac{\alpha^4}{4!} [1.9466\alpha - 0.3533\alpha^3 + 0.01996\alpha^5 - \ldots]$$

$$P_3(\alpha) = -\frac{\alpha^6}{6!} [1.1232\alpha^2 - 0.1068\alpha^4 + \ldots]$$

$$Q_3(\alpha) = -\frac{\alpha^6}{6!} [2.1199\alpha - 0.3858\alpha^3 + \ldots]$$

$$P_4(\alpha) = \frac{\alpha^8}{8!} [1.2816\alpha^2 - \ldots], \qquad Q_4(\alpha) = \frac{\alpha^8}{8!} [2.3946\alpha - \ldots]$$

The first approximation to the solution of equation (4.1) consequently takes the form

$$H_0^{(1)}(x) = \frac{2W_0}{\pi(1-\nu)} \left\{1 - \frac{2}{\pi(1-\nu)} \sum_{m=0}^{\infty} [P_m(\alpha) + iQ_m(\alpha)] M_m(x)\right\} \tag{4.9}$$

The second term in the curly brackets in equation (4.9) is the dynamic correction to the solution of the corresponding static problem.

We consider now equation (4.2) with the kernel (4.4) and seek the solution in the first approximation. Calculating the integral

$$\int_0^1 K_1(x, y)\, y dy = \int_0^1 y^2 \left(\frac{1-x^2}{1-y^2}\right)^{1/2} \left\{\frac{d}{dx}\int_0^1 \frac{dt}{\sqrt{t^2-x^2}} \int_0^\infty G(\sigma)\, J_1(\alpha y\sigma) \sin(\alpha t\sigma)\, d\sigma\right\} dy$$

and changing the order of integration, we obtain

$$\int_0^1 K_1(x, y)\, y dy = \sqrt{1-x^2}\frac{d}{dx}\int_x^1 \frac{dt}{\sqrt{t^2-x^2}}\int_0^\infty G(\sigma) \sin(\alpha t\sigma)\left(\frac{\sin\alpha\sigma}{(\alpha\sigma)^2} - \frac{\cos\alpha\sigma}{\alpha\sigma}\right) d\sigma \quad (4.10)$$

Applying the theory of residues to the calculation of the inner integral

$$I_1(\alpha, t) = \int_0^\infty G(\sigma)\left(\frac{\sin\alpha\sigma}{(\alpha\sigma)^2} - \frac{\cos\alpha\sigma}{\alpha\sigma}\right)\sin(\alpha t\sigma)\, d\sigma$$

we find

$$I_1(\alpha, t) = \left[\pi c\left(\sin\alpha\sigma_0 - \frac{1-\cos\alpha\sigma_0}{\alpha\sigma_0}\right)\sin(\alpha\sigma_0 t) + \right.$$
$$\left. + \int_0^1 V(\sigma)\left(\sin\alpha\sigma - \frac{1-\cos\alpha\sigma}{\alpha\sigma}\right)\sin(\alpha t\sigma)\, d\sigma\right] +$$
$$+ i\left[\pi c\left(\cos\alpha\sigma_0 - \frac{\sin\alpha\sigma_0}{\alpha\sigma_0}\right)\sin(\alpha\sigma_0 t) + \int_0^1 V(\sigma)\left(\cos\sigma - \frac{\sin\alpha\sigma}{\alpha\sigma}\right)\sin(\alpha t\sigma)\, d\sigma\right]$$

Expanding now the function $I_1(\alpha t)$ in powers of t and the α-dependent coefficients of the resultant series in powers of α, we obtain

$$I_1(\alpha, t) = \sum_{m=1}^\infty [\mathring{P}_m(\alpha) + i\mathring{Q}_m(\alpha)]\, t^{2m-1} \quad (4.11)$$

$$\mathring{P}_m(\alpha) = \frac{(-1)^{m-1}\alpha^{2m-1}}{(2m-1)!}\sum_{k=1}^\infty c_k^{\ m}\alpha^{2k-1}, \quad \mathring{Q}_m(\alpha) = \frac{(-1)^{m-1}\alpha^{2m-1}}{(2m-1)!}\sum_{k=1}^\infty d_k^{(m)}\alpha^{2k-2}$$

$$c_k^{(m)} = \frac{(-1)^{k-1}}{(2k-2)!\,2k}(\pi c\sigma_0^{2m+2k-2} + q_{2m+2k-2}) \quad (k \neq 1)$$

$$d_k^{(m)} = \frac{(-1)^{k-1}}{(2k-3)!\,(2k-1)}(\pi c\sigma_0^{2m+2k-3} + q_{2m+2k-3}), \quad d_1^{(m)} = 0$$

Substituting for $I_1(\alpha, t)$ in (4.10) its expansion (4.11) and integrating, we obtain

$$\int_0^1 K_1(x, y)\, y dy = \sum_{m=1}^\infty [\mathring{P}_m(\alpha) + i\mathring{Q}_m(\alpha)] M_{m-1}(x)$$

We quote the expansions of the first few coefficients $\mathring{P}_m(\alpha)$ and $\mathring{Q}_m(\alpha)$ for $h^2 = 1/3$ $(\nu = 1/4)$

$$P_1^\circ(\alpha) = \frac{\alpha}{1!}[1.0068\alpha - 0.2374\alpha^3 + 0.01331\alpha^5 - 0.0003900\alpha^7 + \ldots]$$

$$Q_1^\circ(\alpha) = \frac{\alpha}{1!}[-0.6371\alpha^2 + 0.06489\alpha^4 - 0.002524\alpha^6 + 0.00005279\alpha^8 - \ldots]$$

$$P_2^\circ(\alpha) = \frac{-\alpha^3}{3!}[0.9495\alpha - 0.2398\alpha^3 + 0.01560\alpha^5 - 0.0004450\alpha^7 + \ldots]$$

$$Q_2^\circ(\alpha) = \frac{-\alpha^3}{3!}[-0.6489\alpha^2 + 0.07066\alpha^4 - 0.002851\alpha^6 + \ldots]$$

$$P_3^\circ(\alpha) = \frac{\alpha^5}{5!}[0.9583\alpha - 0.2808\alpha^3 + 0.01780\alpha^5 - \ldots]$$

$$Q_3^\circ(\alpha) = \frac{\alpha^5}{5!}[-0.7066\alpha^2 + 0.07982\alpha^4 - \ldots]$$

$$P_4^\circ(\alpha) = \frac{-\alpha^7}{7!}[1.1232\alpha - 0.3204\alpha^3 + \ldots], \qquad Q_4^\circ(\alpha) = \frac{-\alpha^7}{7!}[-0.7982\alpha^2 + \ldots]$$

The first approximation to the solution of equation (4.2) consequently takes the form

$$H_1^{(1)}(x) = \frac{8\alpha\beta}{\pi(1-\nu)}\left\{1 - \frac{2}{\pi(1-\nu)}\sum_{m=1}^{\infty}[P_m^\circ(\alpha) + iQ_m^\circ(\alpha)]\, M_{m-1}(x)\right\} x$$

Here, as in (4.9), the second term is the dynamic correction to the solution of the corresponding static problem. We note that in obtaining the subsequent approximations, the same integrals will be encountered, except in place of $\sin \alpha\sigma/\alpha\sigma$ in the integrals $I(\alpha, t)$ and $I_1(\alpha, t)$, there will be $J_{m+1/2}(\alpha\sigma)/(\alpha\sigma)^{m+1/2}$.

Thus in principle the subsequent approximations are found in a similar manner, but the amount of calculation increases considerably.

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